RC Time Circuits Explained

By: Ethan Hardie

RC circuits are undoubtedly one of the most fundamental systems in electronics. For essentially all time related electronic circuits, capacitors and resistors are often found together relying on each other to serve a purpose. And, while they work extremely simply on their own, things start to get a lot more difficult when they start interacting with each other.

So, in order to understand the true physics of RC timing, we first need to gain a fundamental grasp of the formulas that govern the behaviour of both capacitors and resistors.

The ability for any capacitor to hold charge is given by the equation $C = \frac{Q}{V}$, $where\ C = Capacitance,\ Q = Charge,\ V = Voltage$. In other words, 1 Farad (the unit for capacitance) is equivalent to one Coulomb (the unit for charge) per Volt (the unit for electrical potential). This denotes that for every volt gained, the capacitor receives enough potential difference to retain an additional coulomb of charge. Now, before we understand RC circuits, let's connect a theoretical capacitor of 1mF to an ideal 50mA constant current source. After 1 second, what will be the voltage of the capacitor?

If we work in differentials (tiny changes), we can rewrite the original formula as $C(t) = \frac{dQ(t)}{dV(t)}$, which can be rearranged to $dQ = CdV$. In this case, the changing variables with respect to time are both charge (Q), and voltage (V).

If we take the formula for electric current (which is $I = \frac{Q}{t}$), and work in the same way, we get $I(t) = \frac{dQ}{dt}$, which represents the rate of change of charge with respect to time. Since we defined dQ equal to CdV in the formula for capacitance, we can substitute the variable dQ into the electric current equation for a complete equation ($I(t) = \frac{CdV}{dt}$). This separable differential equation provides the apparent current through the capacitor over a function of time.

Solution for V:

$$I(t) = C\frac{dV}{dt}$$

$$\frac{dV}{dt} = \frac{I(t)}{C}$$

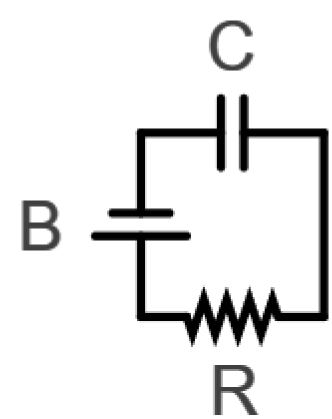
$$\int \frac{dV}{dt} dt = \int \frac{I(t)}{C} dt$$

$$\int dv = \frac{1}{C} \int I(t)dt$$

$$V(t) = \frac{1}{C} \int I(t)dt$$

When the equation is rearranged to solve for voltage, if becomes: $V(t) = 1/c \int I(t)dt$. Since charge is the total amount of current passing through a point over a certain amount of time, it is represented here as the infinite sum of currents passing through the point with respect to time.

This formula is essential to any calculation regarding capacitance and is also important in gaining an understanding of other formulas where capacitors interact with other passive circuit elements. To revisit the original question, the capacitor will receive a voltage of $\frac{1}{C} \int I_C dt = \frac{I_C t}{C} = \frac{(50mA)\cdot(1s)}{(1mF)} = 50V$. Note that since the current is constant and does not vary as a function of time, we can simply integrate by multiplying by t. This means for any constant values, the equation simply returns a linear function.



This function works well for any current source, and that's great, it's what it's designed to do. The problem is that the equation doesn't allow us to deal with multiple circuit elements. If a resistor and battery were instead introduced to this circuit, its current would be modelled by Ohm's famous equation ($V = IR$). This is a problem because its current limit depends on the voltage drop across it. Since the voltage of the capacitor changes with time, it will impact the voltage drop across the resistor and thus its current output. In our example circuit, as the capacitor charges, the voltage potential across the resistor decreases, and thus the current into the capacitor does as well. This indicates that the charge function will follow a non-linear curve, as the slope of the graph depends on the remaining voltage difference between the source and the capacitor.

In order to find a relationship between the voltage of the capacitor, initial battery voltage and resistance value, we can use Kirchhoff's Voltage Law (KVL) on our simple circuit model.

According to KVL, we know that the algebraic sum of all voltage differences around any closed loop is zero. From this, we can construct the following equation (assuming the capacitor is uncharged):

$$V_B - V_C - V_R = 0;$$

$$Where\ V_B = Battery\ Voltage,\ V_C = Capacitor\ Voltage,\ V_R = Resistor\ Voltage$$

Next, we will solve for the voltage across the capacitor as a function of time. We'll call this function $V_C(t)$.

$$V_B - V_C(t) - V_R = 0$$

We know that the voltage across the resistor is just equal to its current times the resistance:

$$V_B - V_C(t) - IR = 0$$

And we know that the current through the resistor is equal to the current through the capacitor (due to kirchhoff's current law). Since the formula for electric current through the circuit is equal to dQ/dt, and dQ is equal to CdV, it can be inferred that:

$$V_B - V_C(t) - (C\frac{dV}{dt})R = 0.$$

Rearranging the equation into standard form for separable differential equations, we get:

$$RC\frac{dV}{dt} + V_C(t) = V_B$$

A first order ODE is called homogeneous if it can be written in the form:

$$\frac{dy}{dx} + P(x)y = 0$$

However in this case the equation is non-homogenous, meaning that the solution will involve a particular solution due to the presence of a non-zero right hand term (V_B). In order to solve this equation, we must first solve the homogeneous part of the equation:

$$RC\frac{dV}{dt} + V_C(t) = 0$$

We can perform a basic algebraic manipulation by dividing both sides by RC:

$$\frac{dV}{dt} + \frac{V_C(t)}{RC} = 0$$

$\frac{1}{V_C(t)} \cdot \frac{dV}{dt} = -\frac{1}{RC}$

Now that the equation is in the form $N(y)\frac{dy}{dx} = M(x)$, with all constants on the right side. we can integrate both sides:

$\int \frac{1}{V_C(t)} \cdot \frac{dV}{dt} dt = \int -\frac{1}{RC} dt$

$\int \frac{1}{V_C(t)} dV = \int -\frac{1}{RC} dt$

$ln\left|V_C(t)\right| + C_1 = -\frac{t}{RC} + C_2$

Both constants can combine to form one, because $C_1 - C_2 = k$

$ln\left|V_C(t)\right| = -\frac{t}{RC} + k$

To find $V_C(t)$, we need to exponentiate both sides with the base e.

$V_C(t) = e^{-\frac{t}{RC}+k}$

Recall that the sum of powers is equal to the multiplication of powers with the same base. Thus:

$e^{-\frac{t}{RC}+k} = e^{-\frac{t}{RC}} \cdot e^k$

Since e is a constant (~ 2.71828), e^k is also constant. Thus:

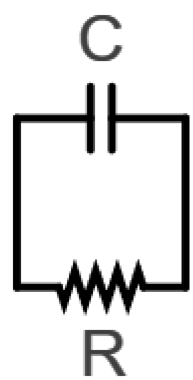
$V_C(t) = ke^{-\frac{t}{RC}}$

If we set time to $t = 0$, the function collapses to:

$V_C(0) = ke^0$

$V_C(0) = k$

This suggests that the constant, "k" represents the initial capacitor voltage. Let's call it V_0:



$V_C(t) = V_0 e^{-\frac{t}{RC}}$

What is left is simply an exponential decay function, which demonstrates the natural discharge rate of a capacitor. In fact, this models the circuit if we were to exclude the battery.

The particular, or non-homogenous solution allows us to include a "forcing function," that represents an external input drawing the function to a certain value. In this case it is the battery, which continuously adds energy into the system and determines what the function will approach after infinite time passes. Our original equation models the circuit with no external inputs or forces; simply an initial voltage. By solving the particular solution, we can find a function that models the external forces of the circuit which can then be combined with the homogeneous solution to find a general equation. To solve the particular component, we can rewrite the full equation:

$$RC\frac{dV}{dt} + V_C(t) = V_B$$

We know that V_B is practically constant (as battery voltage doesn't change much in regard to the size of the capacitor), so we can assume:

$$V_C(t) = k$$

Therefore we can ignore the derivative and we are left with:

$$V_C(t) = V_B$$

Usually the particular solution is much more difficult than this, but since we are dealing with a constant and not a function of x, it's much simpler. To obtain the final general solution, we add both solutions together:

$$V_C(t) = V_h(t) + V_p(t)$$

$$V_C(t) = V_0 e^{-\frac{t}{RC}} + V_B$$

At time $t = 0$, the equation turns into $0 = V_0 + V_B$ ($V_C(t) = 0$ because, at $t = 0$, the capacitor is uncharged), and thus algebraically $V_0 =- V_B$ (due to KVL).

This means that we can factor the variables out of the equation and we will be left with the final equation:

$$V_C(t) = V_B(1 - e^{-\frac{t}{RC}}).$$

However, graphing this function yields a function that never quite reaches the original charging voltage, and this is true! In an ideal world, capacitors never truly reach their maximum voltage. This is where the practicality of a time constant reveals itself. The equation for an RC time constant is:

$$\tau = RC$$

The time constant is represented by the Greek letter *tau*, and represents the amount of time it takes for the capacitor to reach a voltage potential of approximately 63.2% ($1 - e^{-1}$) of the supply. To understand why we use this method of measurement, let's find what $V_C(t)$ is equal to if we set $t = RC$.

$V_C(t) = V_B(1 - e^{-\frac{t}{RC}})$

$V_C(t) = V_B(1 - e^{-\frac{t}{\tau}})$

The function simplifies to:

$V_C(t) = V_B(1 - e^{-1})$

Which can be further reduced to:

$V_C(t) \approx V_B(0.63212055882)$

And this is where the **63.2%** number comes from. It's the voltage that the capacitor reaches after RC seconds have passed. The time constant is more often used in rough physics calculations because it is much easier to use simple formulas than to solve exponents when working on large circuits. In theory, a capacitor charging in a simple RC (resistor-capacitor) circuit approaches its maximum voltage asymptotically. This means that it never truly reaches the full charging voltage in finite time but gets closer and closer as time progresses. However, a general rule of thumb is that after 5 time constants, the capacitor is considered effectively charged, as it reaches about 99.3% of the charging voltage.

Now these equations can be applied to all sorts of situations involving modelling timing circuits, but I would encourage you to also physically explore the behaviour of capacitors and resistors. I find that it can be very easy to get caught up in all of the maths without truly conceptualising what the formulas mean. After all, these equations are found almost everywhere in electronics, so it's essential you get them under your belt. Thanks for watching.